2022

AP® Calculus BC

Sample Student Responses and Scoring Commentary

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Free-Response Question 1

- ☑ Scoring Guidelines
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Part A (AB or BC): Graphing calculator required

Question 1 — 9 points

General Scoring Notes
The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

From 5 A.M. to 10 A.M., the rate at which vehicles arrive at a certain toll plaza is given by $A(t) = 450\sqrt{\sin(0.62t)}$, where t is the number of hours after 5 A.M. and $A(t)$ is measured in vehicles per hour. Traffic is flowing smoothly at 5 A.M. with no vehicles waiting in line.

Model Solution	Scoring	

(a) Write, but do not evaluate, an integral expression that gives the total number of vehicles that arrive at the toll plaza from 6 A.M. ($t = 1$) to 10 A.M. ($t = 5$).

The total number of vehicles that arrive at the toll plaza from 6 A.M. to 10 A.M. is given by $\int_1^5 A(t)\,dt$.	Answer	**1 point**

Scoring notes:

- The response must be a definite integral with correct lower and upper limits to earn this point.
- Because $|A(t)| = A(t)$ for $1 \le t \le 5$, a response of $\int_1^5 \left|450\sqrt{\sin(0.62t)}\right| dt$ or $\int_1^5 |A(t)|\,dt$ earns the point.
- A response missing dt or using dx is eligible to earn the point.
- A response with a copy error in the expression for $A(t)$ will earn the point only in the presence of $\int_1^5 A(t)\,dt$.

Total for part (a) 1 point

(b) Find the average value of the rate, in vehicles per hour, at which vehicles arrive at the toll plaza from 6 A.M. ($t = 1$) to 10 A.M. ($t = 5$).

Average $= \frac{1}{5-1}\int_1^5 A(t)\,dt = 375.536966$ The average rate at which vehicles arrive at the toll plaza from 6 A.M. to 10 A.M. is 375.537 (or 375.536) vehicles per hour.	Uses average value formula: $\frac{1}{b-a}\int_a^b A(t)\,dt$	**1 point**
	Answer	**1 point**

Scoring notes:

- The use of the average value formula, indicating that $a = 1$ and $b = 5$, can be presented in single or multiple steps to earn the first point. For example, the following response earns both points: $\int_1^5 A(t)\,dt = 1502.147865$, so the average value is 375.536966.
- A response that presents a correct integral along with the correct average value, but provides incorrect or incomplete communication, earns 1 out of 2 points. For example, the following response earns 1 out of 2 points: $\int_1^5 A(t)\,dt = 1502.147865 = 375.536966$.
- The answer must be correct to three decimal places. For example, $\frac{1}{5-1}\int_1^5 A(t)\,dt = 375.536966 \approx 376$ earns only the first point.
- Degree mode: A response that presents answers obtained by using a calculator in degree mode does not earn the first point it would have otherwise earned. The response is generally eligible for all subsequent points (unless no answer is possible in degree mode or the question is made simpler by using degree mode). In degree mode, $\frac{1}{4}\int_1^5 A(t)\,dt = 79.416068$.
- Special case: $\frac{1}{5}\int_1^5 A(t)\,dt = 300.429573$ earns 1 out of 2 points.

Total for part (b) 2 points

(c) Is the rate at which vehicles arrive at the toll plaza at 6 A.M. ($t = 1$) increasing or decreasing? Give a reason for your answer.

$A'(1) = 148.947272$

Because $A'(1) > 0$, the rate at which the vehicles arrive at the toll plaza is increasing.

Considers $A'(1)$	**1 point**
Answer with reason	**1 point**

Scoring notes:

- The response need not present the value of $A'(1)$. The second line of the model solution earns both points.
- An incorrect value assigned to $A'(1)$ earns the first point (but will not earn the second point).
- Without a reference to $t = 1$, the first point is earned by any of the following:
  - 148.947 accurate to the number of decimals presented, with zero up to three decimal places (i.e., 149, 148, 148.9, 148.95, or 148.94)
  - $A'(t) = 148.947$ by itself
- To be eligible for the second point, the first point must be earned.
- To earn the second point, there must be a reference to $t = 1$.
- Degree mode: $A'(1) = 23.404311$

Total for part (c) 2 points

(d) A line forms whenever $A(t) \geq 400$. The number of vehicles in line at time t, for $a \leq t \leq 4$, is given by $N(t) = \int_a^t (A(x) - 400)\, dx$, where a is the time when a line first begins to form. To the nearest whole number, find the greatest number of vehicles in line at the toll plaza in the time interval $a \leq t \leq 4$. Justify your answer.

$N'(t) = A(t) - 400 = 0$

$\Rightarrow A(t) = 400 \Rightarrow t = 1.469372,\ t = 3.597713$

$a = 1.469372$

$b = 3.597713$

Considers $N'(t) = 0$	**1 point**
$t = a$ and $t = b$	**1 point**

t	$N(t) = \int_a^t (A(x) - 400)\, dx$
a	0
b	71.254129
4	62.338346

The greatest number of vehicles in line is 71.

Answer	**1 point**
Justification	**1 point**

Scoring notes:

- It is not necessary to indicate that $A(t) = 400$ to earn the first point, although this statement alone would earn the first point.
- A response of "$A(t) \geq 400$ when $1.469372 \leq t \leq 3.597713$" will earn the first 2 points. A response of "$A(t) \geq 400$" along with the presence of exactly one of the two numbers above will earn the first point, but not the second. A response of "$A(t) \geq 400$" by itself will not earn either of the first 2 points.
- To earn the second point the values for a and b must be accurate to the number of decimals presented, with at least one and up to three decimal places. These may appear only in a candidates table, as limits of integration, or on a number line.
- A response with incorrect notation involving t or x is eligible to earn all 4 points.
- A response that does not earn the first point is still eligible for the remaining 3 points.
- To earn the third point, a response must present the greatest number of vehicles. This point is earned for answers of either 71 or $71.254{*}{*}{*}$ only.
- A correct justification earns the fourth point, even if the third point is not earned because of a decimal presentation error.
- When using a Candidates Test, the response must include the values for $N(a)$, $N(b)$, and $N(4)$ to earn the fourth point. These values must be correct to the number of decimals presented, with up to three decimal places. (Correctly rounded integer values are acceptable.)
- Alternate solution for the third and fourth points:

  For $a \leq t \leq b$, $A(t) \geq 400$. For $b \leq t \leq 4$, $A(t) \leq 400$.

  Thus, $N(t) = \int_a^t (A(x) - 400)\, dx$ is greatest at $t = b$.

  $N(b) = 71.254129$, and the greatest number of vehicles in line is 71.
- Degree mode: The response is only eligible to earn the first point because in degree mode $A(t) < 400$.

Total for part (d) 4 points

Total for question 1 9 points

Answer QUESTION 1 parts (a) and (b) on this page.

Response for question 1(a)

$$\int_1^5 450\sqrt{\sin(0.62t)}\,dt$$

Response for question 1(b)

$$\frac{1}{5-1}\int_1^5 450\sqrt{\sin(0.62t)}\,dt = 375.537 \text{ vehicles per hour}$$

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Answer QUESTION 1 parts (c) and (d) on this page.

Response for question 1(c)

$A'(1) = 148.947$ vehicles per hour per hour

The rate at which vehicles arrive at the toll plaza at 6 A.M. $(t=1)$ is increasing because the rate of change of $A(t)$ at $t=1$ is positive.

Response for question 1(d)

$N'(t) = A(t) - 400 = 0$ when $t = 3.59771$ hours

$A(t) = 450\sqrt{\sin(0.62t)} = 400$ ~~w=~~ $t = 1.46937$

~~N~~ t	$N(t)$
1.46937	0
3.59771	71.2541
4	62.3383

The greatest number of vehicles in line is 71 vehicles at $t = 3.59771$ hours because $N(t)$ achieves a relative maximum at $t = 3.59771$, and since $t = 3.59771$ is the only critical number on the given interval, the relative maximum is the absolute maximum.

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Answer QUESTION 1 parts (a) and (b) on this page.

Response for question 1(a)

$$\int_1^5 A(t)\,dt$$

$$A(t) = 450\sqrt{\sin(0.62t)}$$

Response for question 1(b)

$$\frac{1}{5-1}\int_1^5 A(t)\,dt$$

$$= 375{,}537\ \frac{\text{vehicles}}{\text{hour}}$$

the

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Answer QUESTION 1 parts (c) and (d) on this page.

Response for question 1(c)

$A(t) = 450\sqrt{\sin(0.62t)}$

$A'(1) = 148.947\ldots$

At 6 AM or $\overset{\text{time}}{t=1}$, the rate at which vehicles arrive @ the toll plaza is increasing because $A'(1) > 0$.

Response for question 1(d)

Abs. max of # vehicles.

$a \leq t \leq 4$

$N(t) = \int_a^t (A(t) - 400)\,dt$

$N'(t) = A(t) - 400$

$N'(t) = 0$ or DNE

$450\sqrt{\sin(0.62t)} = 400$

$\sqrt{\sin(0.62t)} = \frac{8}{9}$

$\sin(0.62t) = \frac{64}{81}$

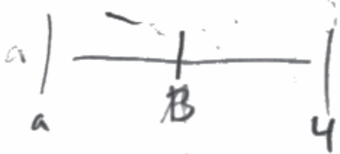
$t = \dfrac{\arcsin\left(\frac{64}{81}\right)}{0.62}$

$t = 1.469\ldots$

$t \to B$

$\sqrt{\sin(0.62t)} = 1$

$t = \boxed{1.612}$



vehicles = $450\sqrt{\sin(0.62t)}$

if $\sqrt{\sin(0.62t)} = 1$, # vehicles $= 450$

Answer QUESTION 1 parts (a) and (b) on this page.

Response for question 1(a)

$$\int_1^5 A(t)\, dx$$

$$\int_1^5 450\sqrt{\sin(.62t)}\, dt$$

Response for question 1(b)

$$\cancel{\frac{A(5) - A(1)}{5-1}} =$$

$$\frac{1}{5}\int_1^5 A'(t)\, dt$$

81.0498

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Answer QUESTION 1 parts (c) and (d) on this page.

Response for question 1(c)

increasing because $A'(t)$ is positive @ $t=1$ ∴ the rate would be increasing

$A'(1) = 148.447$

Response for question 1(d)

$0 = \int_0^t (A(x) - 400)\, dx$

Question 1

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

The context of this problem is vehicles arriving at a toll plaza at a rate of $A(t) = 450\sqrt{\sin(0.62t)}$ vehicles per hour, with time t measured in hours after 5 A.M., when there are no vehicles in line.

In part (a) students were asked to write an integral expression that gives the total number of vehicles that arrive at the plaza from time $t = 1$ to time $t = 5$. A correct response would report $\int_1^5 A(t)\,dt$.

In part (b) students were asked to find the average value of the rate of vehicles arriving at the toll plaza over the same time interval, $t = 1$ to $t = 5$. A correct response would report $\frac{1}{4}\int_1^5 A(t)\,dt$ and then evaluate this definite integral using a calculator to find an average value of 375.537. (The units, vehicles per hour, were given in the statement of the problem.)

In part (c) students were asked to reason whether the rate of vehicles arriving at the toll plaza is increasing or decreasing at 6 A.M., when $t = 1$. A correct response would use a calculator to determine that $A'(1)$, the derivative of the function $A(t)$ at this time, is positive ($A'(1) = 148.947$) and conclude that because $A'(1)$ is positive, the rate of vehicles arriving at the toll plaza is increasing.

Finally, in part (d) students were told that a line of vehicles forms when $A(t) \geq 400$ and the number of vehicles in line is given by the function $N(t) = \int_a^t (A(x) - 400)\,dx$, where a denotes the time, $a \leq t \leq 4$, when the line first begins to form. Students were asked to find the greatest number of vehicles in line at the plaza, to the nearest whole number, in the time interval $a \leq t \leq 4$ and to justify their answer. A correct response would recognize that the greatest number of vehicles is the maximum value of $N(t)$ on the closed interval $a \leq t \leq 4$. To find this maximum, a response should first determine the times t, $0 < t \leq 4$, when the derivative of $N(t)$ is 0. This requires using the Fundamental Theorem of Calculus to find $N'(t) = A(t) - 400$ and then using a calculator to determine that $N'(t)$ is equal to zero when $t = a = 1.469372$ and when $t = b = 3.597713$. A response should then evaluate the function $N(t)$ at each of the values $t = a$, $t = b$, and $t = 4$ to determine that the greatest number of vehicles in line is $N(b) = 71$.

Sample: 1A
Score: 9

The response earned 9 points: 1 point in part (a), 2 points in part (b), 2 points in part (c), and 4 points in part (d).

In part (a) the response earned the point with the definite integral presented.

In part (b) the response earned the first point for the average value expression on the left side of the given equation. The second point was earned for the number on the right side of the equation, which is correct to three decimal places.

In part (c) the response earned the first point for the left side of the equation in the first line. The second point was earned with the concluding sentence.

Question 1 (continued)

In part (d) the response earned the first point with the equation on the left side of line 1. The middle expression, "$A(t) - 400,$" of the equation is not needed to earn that point. The second point was earned with the values at the end of line 1 and line 2. The third point was earned in line 2 of the sentence on the right by identifying $71.$ The Candidates Test table on the left side earned the fourth point. The sentence on the right is consistent with the information in the Candidates Test table.

Sample: 1B
Score: 6

The response earned 6 points: 1 point in part (a), 2 points in part (b), 2 points in part (c), and 1 point in part (d).

In part (a) the response earned the point with the definite integral presented in line 1.

In part (b) the response earned the first point for the expression in line 1. The second point was earned for the correct value in line 2.

In part (c) the response earned the first point for the left side of the equation in line 2. The second point was earned with the concluding sentence.

In part (d) the response earned the first point with the equation on the left in line 3. The second point was not earned because the value of 3.598 is never given. The third point was not earned because there is no value given for $N(3.598).$ The fourth point was not earned because no justification is presented.

Sample: 1C
Score: 3

The response earned 3 points: 1 point in part (a), no points in part (b), 2 points in part (c), and no points in part (d).

In part (a) the response earned the point with the definite integral presented in line 1. The definite integral in line 2 is not necessary.

In part (b) the response did not earn the first point because the integrand given in line 1 after the crossed-out work is $A'(t)$ and not $A(t).$ Because the integrand presented is $A'(t)$ the response is not eligible to earn the second point.

In part (c) the response earned the first point with the statement "$A'(t)$ is positive @ $t = 1$" in line 1. The second point was earned in line 1 with the prior words: "increasing because $A'(t)$ is positive @ $t = 1.$"

In part (d) the response did not earn any points because no correct work is presented.

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AP® Calculus BC

Sample Student Responses and Scoring Commentary

Inside:

Free-Response Question 2

☑ Scoring Guidelines

☑ Student Samples

☑ Scoring Commentary

Part A (BC): Graphing calculator required

Question 2 — 9 points

General Scoring Notes

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

A particle moving along a curve in the xy-plane is at position $(x(t), y(t))$ at time $t > 0$. The particle moves in such a way that $\frac{dx}{dt} = \sqrt{1+t^2}$ and $\frac{dy}{dt} = \ln\left(2+t^2\right)$. At time $t = 4$, the particle is at the point $(1, 5)$.

	Model Solution	Scoring	
(a)	Find the slope of the line tangent to the path of the particle at time $t = 4$.		
	$\left.\frac{dy}{dx}\right\|_{t=4} = \frac{y'(4)}{x'(4)} = \frac{\ln 18}{\sqrt{17}} = 0.701018$ The slope of the line tangent to the path of the particle at time $t = 4$ is 0.701.	Answer	**1 point**

Scoring notes:

- To earn the point, the setup used to perform the calculation must be evident in the response. The following examples earn the point: $\frac{y'(4)}{x'(4)} = 0.701$, $\frac{\ln\left(2+4^2\right)}{\sqrt{1+4^2}}$, or $\frac{\ln 18}{\sqrt{17}}$.
- Note: A response with an incorrect equation of the form "function = constant", such as $\frac{y'(t)}{x'(t)} = \frac{\ln(18)}{\sqrt{17}}$, will not earn the point. However, such a response will be eligible for any points for similar errors in subsequent parts.

Total for part (a) 1 point

(b) Find the speed of the particle at time $t = 4,$ and find the acceleration vector of the particle at time $t = 4.$

$\sqrt{(x'(4))^2 + (y'(4))^2} = \sqrt{17 + (\ln 18)^2} = 5.035300$ The speed of the particle at time $t = 4$ is $5.035.$	Speed	**1 point**
$a(4) = \langle x''(4), y''(4) \rangle = \left\langle \frac{4}{\sqrt{17}}, \frac{4}{9} \right\rangle = \langle 0.970143, 0.444444 \rangle$ The acceleration vector of the particle at time $t = 4$ is $\langle 0.970, 0.444 \rangle.$	First component of acceleration	**1 point**
	Second component of acceleration	**1 point**

Scoring notes:

- To earn any of these points, the setup used to perform the calculation must be evident in the response. For example, $\sqrt{(x'(4))^2 + (y'(4))^2} = 5.035$ or $\sqrt{17 + (\ln 18)^2}$ earns the first point, and $\langle x''(4), y''(4) \rangle = \left\langle \frac{4}{\sqrt{17}}, \frac{4}{9} \right\rangle$ earns both the second and third points.
- The second and third points can be earned independently.
- If the acceleration vector is not presented as an ordered pair, the x- and y-components must be labeled.
- If the components of the acceleration vector are reversed, the response does not earn either of the last 2 points.
- A response which correctly calculates expressions for both $x''(t) = \frac{t}{\sqrt{1 + t^2}}$ and $y''(t) = \frac{2t}{2 + t^2},$ but which fails to evaluate both of these expressions at $t = 4,$ earns only 1 of the last 2 points.
- An unsupported acceleration vector earns only 1 of the last 2 points.

Total for part (b) 3 points

(c) Find the y-coordinate of the particle's position at time $t = 6$.

$y(6) = y(4) + \int_4^6 \ln\left(2 + t^2\right) dt$	Integrand	**1 point**
	Uses $y(4)$	**1 point**
$= 5 + 6.570517 = 11.570517$ The y-coordinate of the particle's position at time $t = 6$ is 11.571 (or 11.570).	Answer	**1 point**

Scoring notes:

- For the first point, an integrand of $\ln\left(2 + t^2\right)$ can appear in either an indefinite integral or an incorrect definite integral.
- A definite integral with incorrect limits is not eligible for the answer point.
- Similarly, an indefinite integral is not eligible for the answer point.
- For the second point, the value for $y(4)$ must be added to a definite integral.
- A response that reports the correct x-coordinate of the particle's position at time $t = 6$ as $x(6) = x(4) + \int_4^6 \sqrt{1 + t^2}\, dt = 11.200$ (or 11.201) instead of the y-coordinate, earns 2 out of the 3 points.
- A response that earns the first point but not the second can earn the third point with an answer of 6.571 (or 6.570).
- If the differential is missing:
  - ○ $y(6) = \int_4^6 \ln\left(2 + t^2\right)$ earns the first point and is eligible for the third.
  - ○ $y(6) = \int_4^6 \ln\left(2 + t^2\right) + y(4)$ does not earn the first point but is eligible for the second and third points in the presence of the correct answer.
  - ○ $y(6) = y(4) + \int_4^6 \ln\left(2 + t^2\right)$ earns the first two points and is eligible for the third.

Total for part (c) 3 points

(d) Find the total distance the particle travels along the curve from time $t = 4$ to time $t = 6$.

$\int_4^6 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$	Integrand	**1 point**
$= 12.136228$ The total distance the particle travels along the curve from time $t = 4$ to time $t = 6$ is 12.136.	Answer	**1 point**

Scoring notes:

- The first point is earned for presenting the correct integrand in a definite integral.
- To earn the second point, a response must have earned the first point and must present the value 12.136.
- An unsupported answer of 12.136 does not earn either point.

Total for part (d) 2 points

Total for question 2 9 points

Answer QUESTION 2 parts (a) and (b) on this page.

Response for question 2(a)

$\frac{dx}{dt} = \sqrt{1+t^2}$

$\frac{dy}{dt} = \ln(2+t^2)$

$\frac{dy}{dx} = \frac{\ln(2+t^2)}{\sqrt{1+t^2}}$

slope of line tangent to particle's path

↳ plug in 4 for t in calc : $\downarrow$ 0.7010

Response for question 2(b)

speed $= \sqrt{(x'(t))^2 + (y'(t))^2}$

↳ $\sqrt{(\sqrt{1+t^2})^2 + (\ln(2+t^2))^2}$

speed of the particle

↳ plug in 4 for t on calculator: $\downarrow$ 5.0353

Acceleration vector

$\left\langle \frac{d}{dt}\left(\sqrt{1+t^2}\right), \frac{d}{dt}\left(\ln(2+t^2)\right)\right\rangle \rightarrow$ plug in 4 $\rightarrow \langle 0.9701, 0.4444\rangle$

2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2

Answer QUESTION 2 parts (c) and (d) on this page.

Response for question 2(c)

at $t=4$, particle is at $(1,5)$

$$5+\int_4^6 \underset{\frac{dy}{dt}}{\underset{\uparrow}{\ln(2+t^2)}}\,dt$$

at time $t=6$, the y-coordinate of the particle's position is

$$\boxed{11.5705}$$

Response for question 2(d)

total distance = magnitude of speed

$$\int_a^b \sqrt{(x'(t))^2+(y'(t))^2}\,dt$$

$$\int_4^6 \sqrt{\left(\sqrt{1+t^2}\right)^2+\left(\ln(2+t^2)\right)^2}\,dt$$

The particle travels

↳ in calculator → 12.1362 from

$t=4$ to $t=6$

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Answer QUESTION 2 parts (a) and (b) on this page.

Response for question 2(a)

$\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ $\frac{\ln(2+t^2)}{\sqrt{1+t^2}}$ at $t=4$

slope = $\boxed{0.701}$

Response for question 2(b)

$\sqrt{(\sqrt{1+t^2})^2 + (\ln(2+t^2))^2}$ = at $t=4$

speed = $\boxed{5.0353}$

$\sqrt{(\sqrt{1+t2})'^2 + (\ln(2+t^2))'^2}$

$(0.970142)^2 + (0.444)^2$ at $t=4$

$= \boxed{1.0671}$ acceleration

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Answer QUESTION 2 parts (c) and (d) on this page.

Response for question 2(c)

~~$\int\sqrt{1+t^2}$~~

$\int_4^6 \ln(2+t^2)\,dt$

at t=4

(1, 5)

↳

$= 6.5705 + 5$

$\boxed{\text{y-coord} = 11.5705}$

Response for question 2(d)

from t=4 to t=6

$\int_4^6 \sqrt{1+t^2}\,dt = 10.2006$

$(1, 5) \rightarrow (11.2006, 11.5705)$

$5.099 \rightarrow 16.1037$

vector difference

$\boxed{11.0047}$

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Answer QUESTION 2 parts (a) and (b) on this page.

Response for question 2(a)

$$\frac{dr}{dt} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\ln(2+t^2)}{\sqrt{1+t^2}} \quad \text{at } t=4$$

$$\frac{\ln(2+16)}{\sqrt{1+16}} = \frac{\ln 18}{\sqrt{17}} = 0.701$$

Response for question 2(b)

$$\frac{dr}{dt} = \frac{dy/dt}{dx/dt} = \frac{\ln(2+t^2)}{\sqrt{1+t^2}} \quad \text{at } t=4 \quad = \underset{\text{speed}}{\underset{\wedge}{0.701}}$$

$$\frac{d^2r}{dt^2} = \frac{\frac{1}{2+t^2}\cdot 2t\cdot\sqrt{1+t^2} - \left(-\frac{1}{\cancel{2}}\cdot\frac{1}{\sqrt{1+t^2}}\cdot\cancel{2}t\right)\left(\ln(2+t^2)\right)}{1+t^2}$$

$$= \frac{\frac{2t\sqrt{1+t^2}}{2+t^2} + \frac{t\ln(2+t^2)}{\sqrt{1+t^2}}}{t^2} = \frac{\frac{8\sqrt{17}}{18} + \frac{4\ln 8}{\sqrt{17}}}{16} = \underset{\text{acceleration}}{\underset{\uparrow}{0.241}}$$

Answer QUESTION 2 parts (c) and (d) on this page.

Response for question 2(c)

$$\frac{dy}{dt} = \frac{\ln(2+t^2)}{1}$$

$$\int 1\ dy = \int \ln(2+t^2)\,dt$$

$$y = \int \ln(2+t^2)\ dt$$

$$\int_4^6 \ln(2+t^2)\ dt$$

$$=$$

Response for question 2(d)

$$\int_4^6 \sqrt{1+\left(\frac{\ln(2+t^2)}{\sqrt{1+t^2}}\right)^2}\ dt = 2.383$$

Question 2

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem a particle is moving along a curve in the xy-plane with position $(x(t), y(t))$. At time $t = 4,$ the particle is at the point $(1, 5)$. The particle moves so that $\frac{dx}{dt} = \sqrt{1 + t^2}$ and $\frac{dy}{dt} = \ln\left(2 + t^2\right)$.

In part (a) students were asked to find the slope of the line tangent to the path of the particle at time $t = 4$. A correct response would provide the setup $\left.\frac{dy}{dx}\right|_{t=4} = \frac{y'(4)}{x'(4)}$ and evaluate to find the slope is $\frac{\ln 18}{\sqrt{17}}$ or, using a calculator, 0.701.

In part (b) students were asked to find the speed of the particle and the acceleration vector of the particle, both at time $t = 4$. A correct response would indicate that the speed of the particle at this time is $\sqrt{(x'(4))^2 + (y'(4))^2} = 5.0353$ and the acceleration is $a(4) = \langle x''(4), y''(4)\rangle = \langle 0.970, 0.444\rangle$. (Either or both of these answers could be provided without use of the calculator as $\sqrt{17 + (\ln 18)^2}$ or $\left\langle \frac{4}{\sqrt{17}}, \frac{4}{9}\right\rangle$, respectively.)

In part (c) students were asked to find the y-coordinate of the particle's position at time $t = 6$. A correct response would integrate the rate of change of the particle's y-position, $\frac{dy}{dt} = \ln\left(2 + t^2\right)$, from time $t = 4$ to time $t = 6$, then add the initial condition $y(4) = 5$ to find a y-coordinate of the particle's position of 11.571.

And in part (d) students were asked to find the total distance the particle travels along the curve from time $t = 4$ to time $t = 6$. A correct response would provide the calculator setup of the integral of the particle's speed over this time interval, then evaluate to find a total distance of 12.136.

Sample: 2A
Score: 9

The response earned 9 points: 1 point in part (a), 3 points in part (b), 3 points in part (c), and 2 points in part (d).

In part (a) the response earned the point with the correct presentation of the function $\frac{dy}{dx}$ in the third line and the correct evaluation of this function at $t = 4$ to produce the answer of 0.7010.

In part (b) the response earned the first point with the correct expression for the speed $\sqrt{(x'(t))^2 + (y'(t))^2}$ and a correct evaluation of this expression at $t = 4$ to produce the answer 5.0353. The second point is earned for the x-component in the last line with the expression $\frac{d}{dt}\left(\sqrt{1 + t^2}\right)$, the statement "plug in 4," and the correct answer of 0.9701. The third point is earned for the y-component in the last line with expression $\frac{d}{dt}\left(\ln\left(2 + t^2\right)\right)$, the statement "plug in 4," and the correct answer of 0.4444.

Question 2 (continued)

In part (c) the response earned the first point by using the expression $\ln\left(2 + t^2\right)$ as an integrand. Adding 5 to the associated definite integral earned the second point, and the circled answer of 11.5705 earned the third point.

In part (d) the response earned the first point with the definite integral in the third line and earned the second point with the correct answer of 12.1362.

Sample: 2B
Score: 5

The response earned 5 points: 1 point in part (a), 1 point in part (b), 3 points in part (c), and no points in part (d).

In part (a) the response earned the point by evaluating the function $\dfrac{\ln\left(2 + t^2\right)}{\sqrt{1 + t^2}}$ at $t = 4$ to produce the correct answer.

In part (b) the first point was earned with the correct equation for speed and the correct evaluation of this equation at $t = 4$ to produce the boxed answer in the second line. The second and third points were not earned because neither the x- nor the y-component of the acceleration vector is provided.

In part (c) the use of $\ln\left(2 + t^2\right)$ as an integrand earned the first point. The phrase "at $t = 4 \rightarrow (1, 5)$" pointing to the $+5$ in the second line earned the second point, and the correct value for the y-coordinate earned the third point.

In part (d) the incorrect integrand did not earn the first point, and the incorrect boxed answer did not earn the second point.

Sample: 2C
Score: 2

The response earned 2 points: 1 point in part (a), no points in part (b), 1 point in part (c), and no points in part (d).

In part (a) the response would have earned the first point with the expression $\dfrac{\ln(2 + 16)}{\sqrt{1 + 16}}$ in the second line. The response simplified the expression correctly and earned the first point.

In part (b) the first point was not earned because the response computes the slope of the tangent line instead of the speed at $t = 4.$ The second and third points were not earned because the response provides neither the x- nor y-component of the acceleration vector.

In part (c) the response earned the first point with the indefinite integral $\int \ln\left(2+t^2\right) dt.$ The second point was not earned because $y(4) = 5$ is not used, and the third point was not earned because the correct answer is not given.

In part (d) the response did not earn the first point because the integrand of the definite integral is incorrect, and the second point was not earned because the answer is incorrect.

2022

AP® CollegeBoard

AP® Calculus BC

Sample Student Responses and Scoring Commentary

Inside:

Free-Response Question 3

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AP Central is the official online home for the AP Program: apcentral.collegeboard.org.

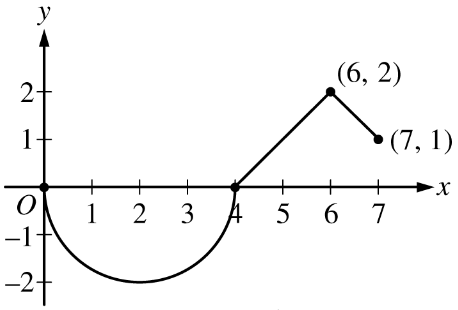
Part B (AB or BC): Graphing calculator not allowed

Question 3 9 points

General Scoring Notes

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.



Graph of f'

Let f be a differentiable function with $f(4) = 3$. On the interval $0 \le x \le 7$, the graph of f', the derivative of f, consists of a semicircle and two line segments, as shown in the figure above.

	Model Solution	Scoring	
(a)	Find $f(0)$ and $f(5)$.		
	$f(0) = f(4) + \int_4^0 f'(x)\,dx = 3 - \int_0^4 f'(x)\,dx = 3 + 2\pi$ $f(5) = f(4) + \int_4^5 f'(x)\,dx = 3 + \frac{1}{2} = \frac{7}{2}$	Area of either region – OR – $\int_0^4 f'(x)\,dx$ – OR – $\int_4^5 f'(x)\,dx$	**1 point**
		$f(0)$	**1 point**
		$f(5)$	**1 point**

Scoring notes:

- A response with answers of only $f(0) = \pm 2\pi$, or only $f(5) = \frac{1}{2}$, or both earns 1 of the 3 points.
- A response displaying $f(5) = \frac{7}{2}$ and a missing or incorrect value for $f(0)$ earns 2 of the 3 points.
- The second and third points can be earned in either order.
- Read unlabeled values from left to right and from top to bottom as $f(0)$ and $f(5)$. A single value must be labeled as either $f(0)$ or $f(5)$ in order to earn any points.

Total for part (a) 3 points

(b) Find the x-coordinates of all points of inflection of the graph of f for $0 < x < 7$. Justify your answer.

The graph of f has a point of inflection at each of $x = 2$ and $x = 6$, because $f'(x)$ changes from decreasing to increasing at $x = 2$ and from increasing to decreasing at $x = 6$.	Answer **1 point** Justification **1 point**

Scoring notes:

- A response that gives only one of $x = 2$ or $x = 6$, along with a correct justification, earns 1 of the 2 points.
- A response that claims that there is a point of inflection at any value other than $x = 2$ or $x = 6$ earns neither point.
- To earn the second point a response must use correct reasoning based on the graph of f'. Examples of correct reasoning include:
  - Correctly discussing the signs of the slopes of the graph of f'
  - Citing $x = 2$ and $x = 6$ as the locations of local extrema on the graph of f'
- Examples of reasoning not (sufficiently) connected to the graph of f' include:
  - Reasoning based on sign changes in f'' unless the connection is made between the sign of f'' and the slopes of the graph of f'
  - Reasoning based only on the concavity of the graph of f
- The second point cannot be earned by use of vague or undefined terms such as “it” or “the function” or “the derivative.”
- Responses that report inflection points as ordered pairs must report the points $(2, 3 + \pi)$ and $(6, 5)$ in order to earn the first point. If the y-coordinates are reported incorrectly, the response remains eligible for the second point.

Total for part (b) 2 points

(c) Let g be the function defined by $g(x) = f(x) - x$. On what intervals, if any, is g decreasing for $0 \le x \le 7$? Show the analysis that leads to your answer.

$g'(x) = f'(x) - 1$	$g'(x) = f'(x) - 1$ **1 point**
$f'(x) - 1 \le 0 \Rightarrow f'(x) \le 1$ The graph of g is decreasing on the interval $0 \le x \le 5$ because $g'(x) \le 0$ on this interval.	Interval with reason **1 point**

Scoring notes:

- The first point can be earned for $f'(x) \le 1$ or the equivalent, in words or symbols.
- Endpoints do not need to be included in the interval to be eligible for the second point.

Total for part (c) 2 points

(d) For the function g defined in part (c), find the absolute minimum value on the interval $0 \le x \le 7$. Justify your answer.

g is continuous, $g'(x) < 0$ for $0 < x < 5$, and $g'(x) > 0$ for $5 < x < 7$. Therefore, the absolute minimum occurs at $x = 5$, and $g(5) = f(5) - 5 = \frac{7}{2} - 5 = -\frac{3}{2}$ is the absolute minimum value of g.	Considers $g'(x) = 0$	**1 point**
	Answer with justification	**1 point**

Scoring notes:

- A justification that uses a local argument, such as "g' changes from negative to positive (or g changes from decreasing to increasing) at $x = 5$" must also state that $x = 5$ is the only critical point.
- If $g'(x) = 0$ (or equivalent) is not declared explicitly, a response that isolates $x = 5$ as the only critical number belonging to $(0, 7)$ earns the first point.
- A response that imports $g'(x) = f'(x)$ from part (c) is eligible for the first point but not the second.
    - In this case, consideration of $x = 4$ as the only critical number on $(0, 7)$ earns the first point.
- Solution using Candidates Test:

  $g'(x) = f'(x) - 1 = 0 \Rightarrow x = 5, x = 7$

x	$g(x)$
0	$3 + 2\pi$
5	$-\frac{3}{2}$
7	$-\frac{1}{2}$

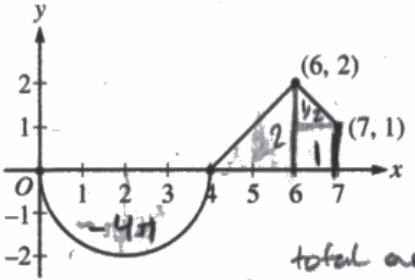
  The absolute minimum value of g on the interval $0 \le x \le 7$ is $-\frac{3}{2}$.

- When using a Candidates Test, a response may import an incorrect value of $f(0) = g(0) > -\frac{3}{2}$ from part (a). The second point can only be earned for an answer of $-\frac{3}{2}$.

Total for part (d) **2 points**

Total for question 3 **9 points**

Answer QUESTION 3 parts (a) and (b) on this page.



Graph of f'

$f(4) = 3$

total area: $3.5 - 4\pi$

Response for question 3(a)

find $f(0)$ & $f(5)$

$\int_4^0 f'(x)\,dx = f(0) - f(4)$

$-\int_0^4 f'(x)\,dx = f(0) - f(4)$

$4\pi = f(0) - 3$

$\boxed{f(0) = 4\pi + 3}$

$\int_4^5 f'(x)\,dx = f(5) - f(4)$

$\frac{1}{2} = f(5) - 3$

$\boxed{f(5) = 3.5}$

Response for question 3(b)

POI: $\boxed{\begin{matrix} x = 2, \\ x = 6 \end{matrix}}$ 6

a point of inflection occurs when the second derivative changes sign. The graph is a graph of f', and f'' is the slope of f', so wherever the graph changes from increasing to decreasing, or vice versa is a point of inflection

Answer QUESTION 3 parts (c) and (d) on this page.

Response for question 3(c)

$g'(x)=f'(x)-1$

$f'(x)-1=0$

$f'(x)=1$

$x=5 \quad x=7$

decreasing in interval $(0,5)$, because $g'(x)=f'(x)-1$ is negative in this interval, meaning g is decreasing.

Response for question 3(d)

crit point: $x=5 \quad x=7$

endpoints: $x=0 \quad x=7$

$g(0)=f(0)-0=4\pi+3$

$g(5)=f(5)-5=3.5-5=-1.5$

$g(7)=f(7)-7=6.5-7=-0.5$

absolute minimum: $\boxed{(5,-1.5)}$

The abs min of $g(x)$ is -1.5 based on the values found through the critical points of the function

$\int_4^7 f'(x)\,dx=f(7)-f(4)$

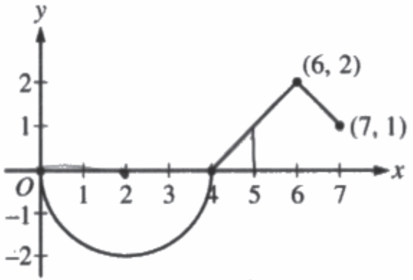
$3.5=f(7)-f(4)$

$3.5=f(7)-3$

$f(7)=6.5$

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Answer QUESTION 3 parts (a) and (b) on this page.



Graph of f'

Response for question 3(a)

$f(4)=3$

using FTC:

$\int_0^4 f'(x) = f(4) - f(0)$

$\int_0^4 f'(x) = \frac{1}{2}\pi(2)^2 = -2\pi$

$-2\pi = 3 - f(0)$

$\boxed{f(0) = 3 + 2\pi}$

$\int_4^5 f'(x) = f(5) - f(4)$

$\int_4^5 f'(x) = \frac{1}{2}(1 \times 1) = \frac{1}{2}$

$\frac{1}{2} = f(5) - 3$

$\boxed{f(5) = 7/2}$

Response for question 3(b)

point of inflection is where $f''(x) = 0$ (horizontal tangent)

There is a horizontal tangent at the bottom of the semicircle, at $\boxed{x = 2}$.

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Answer QUESTION 3 parts (c) and (d) on this page.

Response for question 3(c)

$g(x) = f(x) - x$

$g'(x) = f'(x) - 1$

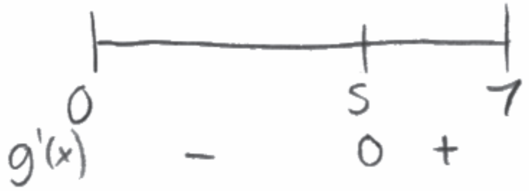
g is decreasing when $g'(x) < 0$

$g'(x) < 0$ when $f'(x) < 1$

$f'(x) < 1$ on the interval $\boxed{0 < x < 5}$

Response for question 3(d)

minimum is where the first derivative goes from negative to positive.

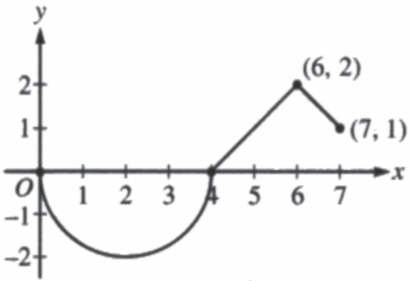


$g'(x)$ goes from negative to positive at $x = 5$

$g(5) = f(5) - (5)$ $g(5) = 7/2 - 5 = \boxed{-3/2 \text{ units}}$

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Answer QUESTION 3 parts (a) and (b) on this page.



Graph of f'

Response for question 3(a)

$f(0) = \int_0^0 f'(x) = 0$ $\quad f(0) = 0$

$f(5) = \int_0^5 f'(x) = \frac{1}{2} - 2\pi$

$f(5) = \frac{1}{2} - 2\pi$

$-\frac{1}{2}\pi r^2$

$-\frac{1}{2} 4\pi$ $\quad -2\pi + \frac{1}{2}$

Response for question 3(b)

$x = 0$

$x = 4$

At $x=0$ and $x=4$, the first derivative switches signs, meaning there is a zero, or a point of inflection for the second derivative.

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Answer QUESTION 3 parts (c) and (d) on this page.

Response for question 3(c)

g is dec on the intervals (0,7) this is because g is negative along the whole graph, making it decreasing on the whole graph.

Response for question 3(d)

$g'(x) = f'(x) - 1$

$f'(x) - 1 = 0$

$x = 5$

abs min at $x = 5$

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Question 3

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem the graph of a function f', which consists of a semicircle and two line segments on the interval $0 \le x \le 7$, is provided. It is also given that this is the graph of the derivative of a differentiable function f with $f(4) = 3$.

In part (a) students were asked to find $f(0)$ and $f(5)$. To find $f(0)$ a correct response uses geometry and the Fundamental Theorem of Calculus to calculate the signed area of the semicircle, $\int_0^4 f'(x)\,dx = -2\pi$, and subtracts this value from the initial condition, $f(4) = 3$, to obtain a value of $3 + 2\pi$. To find $f(5)$ a correct response would add the initial condition to the signed area $\int_4^5 f'(x)dx = \frac{1}{2}$, found using geometry, to obtain a value of $\frac{7}{2}$.

In part (b) students were asked to find the x-coordinates of all points of inflection on the graph of f for $0 < x < 7$ and to justify their answers. A correct response would use the given graph to determine that the graph of $f'(x)$ changes from decreasing to increasing, or vice versa, at the points $x = 2$ and $x = 6$. Therefore, these are the inflection points of the graph of f.

In part (c) students were told that $g(x) = f(x) - x$ and are asked to determine on which intervals, if any, the function g is decreasing. A correct response would find that $g'(x) = f'(x) - 1$ and then use the given graph of f' to determine that when $0 \le x \le 5$, $f'(x) \le 1 \Rightarrow g'(x) \le 0$. Therefore, g is decreasing on the interval $0 \le x \le 5$.

In part (d) students were asked to find the absolute minimum value of $g(x) = f(x) - x$ on the interval $0 \le x \le 7$. A correct response would use the work from part (c) to conclude $g'(x) < 0$ for $0 < x < 5$ and $g'(x) > 0$ for $5 < x < 7$. Thus the absolute minimum of g occurs at $x = 5$. Using the work from part (a), which found the value of $f(5)$, the absolute minimum value of g is $g(5) = f(5) - 5 = \frac{7}{2} - 5 = -\frac{3}{2}$.

Sample: 3A
Score: 8

The response earned 8 points: 2 points in part (a), 2 points in part (b), 2 points in part (c), and 2 points in part (d).

In part (a) the first point was earned for the integral expression to the left of the first equal sign. The second point was not earned because the value of $f(0)$ is reported incorrectly. The third point was earned because the stated value of $f(5)$ is correct.

In part (b) the first point was earned for the boxed answer on lines 1 and 2, which declares the "POI" as $x = 2$ and $x = 6$. The second point was earned for the explanation given in the paragraph below the boxed answer. The response appeals to the change in sign of f'', which alone would not be sufficient but goes on to declare that "f'' is the slope of f'" and indicates correctly that where the graph (of f') "changes from increasing to decreasing, or vice versa is a point of inflection."

Question 3 (continued)

In part (c) the first point was earned on line 1 for $g'(x) = f'(x) - 1.$ The second point was earned on lines 1, 2, and 3 on the right for giving the correct interval $(0, 5)$ with the reason that "$g'(x) = f'(x) - 1$ is negative in this interval."

In part (d) the first point was earned on line 1 for consideration of only $x = 5$ (and the endpoints) as possible locations of the absolute minimum. The second point was earned on line 8 for declaring -1.5 as the absolute minimum. A Candidates Test is carried out correctly, with an incorrect value of $f(0)$ that is greater than $-\frac{3}{2}$ imported from part (a). Such responses are still eligible to earn the second point as long as $g(0)$ is declared to be that same imported value, and there are no mistakes in reported values of $g(5)$ and $g(7)$.

Sample: 3B
Score: 6

The response earned 6 points: 3 points in part (a), no points in part (b), 2 points in part (c), and 1 point in part (d).

In part (a) the response earned the first point for $\int_0^4 f'(x)$ on line 3 on the left. Note that the integrals are missing the differential dx; this oversight is not penalized. The second and third points were earned for the boxed, correct values of $f(0)$ and $f(5)$.

In part (b) the response did not earn the first point, because $x = 6$ is not given among the answers. The second point was not earned because the reasoning that "there is a horizontal tangent at the bottom of the semicircle" is not sufficient. A response must make a specific appeal to f' in order for the second point to be earned.

In part (c) the response earned the first point on line 2 for the correct derivative of $g'(x)$. The response earned the second point for the correct boxed interval on line 5, with correct reasoning on lines 3 and 4.

In part (d) the response earned the first point for consideration of a sign change in $g'(x)$ below the sign chart. Although the response does have the correct answer of $-\frac{3}{2}$, it uses the local argument that "$g'(x)$ goes from negative to positive at $x = 5$" without an appeal to the whole interval $(0, 7)$. Therefore, the second point was not earned. The response appears, perhaps, to address the interval with a sign chart, but the response must explain the conclusions gathered from the chart in order for the point to be earned.

Sample: 3C
Score: 2

The response earned 2 points: 1 point in part (a), no points in part (b), no points in part (c), and 1 point in part (d).

In part (a) the response earned the first point on line 2 for consideration of $\frac{1}{2}$ (the area of the necessary triangular region) to the right of the second equal sign. The response did not earn the second and third points because the answers for $f(0)$ and $f(5)$ are both incorrect.

In part (b) the response did not earn either point because answers other than $x = 2$ or $x = 6$ are given (in this case, $x = 0$ and $x = 4$), which renders the response ineligible for either point.

Question 3 (continued)

In part (c) the response did not earn the first point because $g'(x)$ (or equivalent) is not considered. It is not possible to earn the second point without having earned the first point in part (c).

In part (d) the response earned the first point on line 2 for "$f'(x) - 1 = 0.$" No function value at $x = 5$ is reported; thus, the second point was not earned.

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AP® Calculus BC

Sample Student Responses and Scoring Commentary

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Free-Response Question 4

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Part B (AB or BC): Graphing calculator not allowed

Question 4 **9 points**

General Scoring Notes

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

t (days)	0	3	7	10	12
$r'(t)$ (centimeters per day)	-6.1	-5.0	-4.4	-3.8	-3.5

An ice sculpture melts in such a way that it can be modeled as a cone that maintains a conical shape as it decreases in size. The radius of the base of the cone is given by a twice-differentiable function r, where $r(t)$ is measured in centimeters and t is measured in days. The table above gives selected values of $r'(t)$, the rate of change of the radius, over the time interval $0 \le t \le 12$.

Model Solution	Scoring	

(a) Approximate $r''(8.5)$ using the average rate of change of r' over the interval $7 \le t \le 10$. Show the computations that lead to your answer, and indicate units of measure.

$r''(8.5) \approx \dfrac{r'(10) - r'(7)}{10 - 7} = \dfrac{-3.8 - (-4.4)}{10 - 7}$	$r''(8.5)$ with supporting work	**1 point**
$= \dfrac{0.6}{3} = 0.2$ centimeter per day per day	Units	**1 point**

Scoring notes:

- To earn the first point the supporting work must include at least a difference and a quotient.
- Simplification is not required to earn the first point. If the numerical value is simplified, it must be correct.
- The second point can be earned with an incorrect approximation for $r''(8.5)$ but cannot be earned without some value for $r''(8.5)$ presented.
- Units may be written in any equivalent form (such as $\mathrm{cm}/\mathrm{day}^2$).

Total for part (a) **2 points**

(b) Is there a time t, $0 \le t \le 3$, for which $r'(t) = -6$? Justify your answer.

$r(t)$ is twice-differentiable. $\Rightarrow$ $r'(t)$ is differentiable. $\Rightarrow$ $r'(t)$ is continuous. $r'(0) = -6.1 < -6 < -5.0 = r'(3)$ Therefore, by the Intermediate Value Theorem, there is a time t, $0 < t < 3$, such that $r'(t) = -6$.	$r'(0) < -6 < r'(3)$	**1 point**
	Conclusion using Intermediate Value Theorem	**1 point**

Scoring notes:

- To earn the first point, the response must establish that -6 is between $r'(0)$ and $r'(3)$ (or -6.1 and -5). This statement may be represented symbolically (with or without including one or both endpoints in an inequality) or verbally. A response of "$r'(t) = -6$ because $r'(0) = -6.1$ and $r'(3) = -5$" does not state that -6 is between -6.1 and -5. Thus this response does not earn the first point.
- To earn the second point:
    - The response must state that $r'(t)$ is continuous because $r'(t)$ is differentiable (or because $r(t)$ is twice differentiable).
    - The response must have earned the first point.
        - Exception: A response of "$r'(t) = -6$ because $r'(0) = -6.1$ and $r'(3) = -5$" does not earn the first point because of imprecise communication but may nonetheless earn the second point if all other criteria for the second point are met.
    - The response must conclude that there is a time t such that $r'(t) = -6$. (A statement of "yes" would be sufficient.)
- To earn the second point, the response need not explicitly name the Intermediate Value Theorem, but if a theorem is named, it must be correct.

Total for part (b) 2 points

(c) Use a right Riemann sum with the four subintervals indicated in the table to approximate the value of $\int_0^{12} r'(t)\,dt$.

$\int_0^{12} r'(t)\,dt \approx 3r'(3) + 4r'(7) + 3r'(10) + 2r'(12)$	Form of right Riemann sum	**1 point**
$= 3(-5.0) + 4(-4.4) + 3(-3.8) + 2(-3.5)$ $= -51$	Answer	**1 point**

Scoring notes:

- To earn the first point, at least seven of the eight factors in the Riemann sum must be correct. If there is any error in the Riemann sum, the response does not earn the second point.
- A response of $3(-5.0) + 4(-4.4) + 3(-3.8) + 2(-3.5)$ earns both the first and second points, unless there is a subsequent error in simplification, in which case the response would earn only the first point.
- A response that presents the correct answer, with accompanying work that shows the four products in the Riemann sum (without explicitly showing all of the factors and/or the sum process) does not earn the first point but earns the second point. For example, $-15 + 4(-4.4) + 3(-3.8) + -7$ does not earn the first point but earns the second point. Similarly, $-15,\ -17.6,\ -11.4,\ -7 \rightarrow -51$ does not earn the first point but earns the second point.
- A response that presents the correct answer (-51) with no supporting work earns no points.
- A response that provides a completely correct left Riemann sum and approximation $\int_0^{12} r'(t)\,dt$ (i.e., $3r'(0) + 4r'(3) + 3r'(7) + 2r'(10) = 3(-6.1) + 4(-5.0) + 3(-4.4) + 2(-3.8) = -59.1$) earns 1 of the 2 points. A response that has any error in a left Riemann sum or evaluation for $\int_0^{12} r'(t)\,dt$ earns no points.
- Units are not required or read in this part.

Total for part (c) 2 points

(d) The height of the cone decreases at a rate of 2 centimeters per day. At time $t = 3$ days, the radius is 100 centimeters and the height is 50 centimeters. Find the rate of change of the volume of the cone with respect to time, in cubic centimeters per day, at time $t = 3$ days. (The volume V of a cone with radius r and height h is $V = \frac{1}{3}\pi r^2 h$.)

$\frac{dV}{dt} = \frac{2}{3}\pi rh\frac{dr}{dt} + \frac{1}{3}\pi r^2\frac{dh}{dt}$	Product rule	**1 point**
	Chain rule	**1 point**
$\left.\frac{dV}{dt}\right\|_{t=3} = \frac{2}{3}\pi(100)(50)(-5) + \frac{1}{3}\pi(100)^2(-2) = -\frac{70{,}000\pi}{3}$ The rate of change of the volume of the sculpture at $t = 3$ is approximately $-\frac{70{,}000\pi}{3}$ cubic centimeters per day.	Answer	**1 point**

Scoring notes:

- The first 2 points could be earned in either order.
- A response with a completely correct product rule, missing one or both of the correct differentials, earns the product rule point, but not the chain rule point. For example, $\frac{dV}{dt} = \frac{2}{3}\pi rh + \frac{1}{3}\pi r^2$ earns the first point, but not the second.
- A response that treats r or h (but not both) as a constant is eligible for the chain rule point but not the product rule point. For example, $\frac{dV}{dt} = \frac{2}{3}\pi rh\frac{dr}{dt}$ is correct if h is constant, and thus earns the chain rule point.
- Note: Neither $\frac{dV}{dt} = \frac{2}{3}\pi r\frac{dh}{dt}$ nor $\frac{dV}{dt} = \frac{2}{3}\pi rh\frac{dr}{dt}\frac{dh}{dt}$ earns any points.
- A response that assumes a functional relationship between r and h (such as $r = 2h$), and uses this relationship to create a function for volume in terms of one variable, is eligible for at most the chain rule point. For example, $r = 2h \to V = \frac{1}{3}\pi(2h)^2 h = \frac{4}{3}\pi h^3 \to \frac{dV}{dt} = 4\pi h^2\frac{dh}{dt}$ earns only the chain rule point.
- A response that mishandles the constant $\frac{1}{3}\pi$ cannot earn the third point but is eligible for the first 2 points.
- The third point cannot be earned without both of the first 2 points.
- $\frac{dV}{dt} = \frac{2}{3}\pi(100)(50)(-5) + \frac{1}{3}\pi(100)^2(-2)$ earns all 3 points.
- Units are not required or read in this part.

Total for part (d) 3 points

Total for question 4 9 points

Answer QUESTION 4 parts (a) and (b) on this page.

t (days)	0	3	7	10	12
$r'(t)$ (centimeters per day)	−6.1	−5.0	−4.4	−3.8	−3.5

Response for question 4(a)

$$r''(8.5) \approx \frac{r'(10) - r'(7)}{10-7} = \frac{-3.8 - (-4.4)}{3} = \frac{-3.8+4.4}{3} = \frac{0.6}{3}$$

$$= \boxed{0.2 \text{ cm/day}^2}$$

Response for question 4(b)

Since r is twice-differentiable, r' is also differatiable on the same interval $0 \le t \le 12$. Therefore r' is differentiable between $0 \le t \le 3$ as well since that interval is inside the previous one.

If a function is differentiable, it is also continuous. Therefore, $r'(t)$ is also continuous on $0 \le t \le 3$. According to the Intermediate Value Theorem, any continuous function $f(x)$ on (a,b) will take on every value between $f(a)$ and $f(b)$. Therefore $r'(t)$ will take on every value between -5.0 and -6.1. -6 is inside this range, so there is a time t where $r'(t) = -6$ on $0 \le t \le 3$.

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Answer QUESTION 4 parts (c) and (d) on this page.

Response for question 4(c)

$(3-0)(-5.0) + (7-3)(-4.4) + (10-7)(-3.8) + (12-10)(-3.5)$

$3(-5) + 4(-4.4) + 3(-3.8) + 2(-3.5)$

$-15 + -17.6 - 11.4 - 7$

$-15 - 29 - 7$

$-44 - 7$

$\boxed{-51}$

Response for question 4(d)

$\frac{dh}{dt} = -2 \text{ cm/day}$

$r(3) = 100 \text{ cm}$

$h(3) = 50 \text{ cm}$

$r'(3) = \frac{dr}{dt} = -5.0 \text{ cm/day}$

$V = \frac{1}{3}\pi r^2 h$

$\frac{dV}{dt} = \frac{1}{3}\pi\left[(r^2)\left[\frac{dh}{dt}\right] + (h)\left[2r\frac{dr}{dt}\right]\right]$

$= \frac{1}{3}\pi\left[(100^2)(-2) + (50)(2(100)(-5.0))\right]$

$= \frac{1}{3}\pi\left[-20000 + 50(-1000)\right]$

$= \frac{1}{3}\pi\left[-20000 + -50000\right]$

$= \boxed{\frac{-70000\pi}{3} \text{ cm}^3/\text{day}}$

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Answer QUESTION 4 parts (a) and (b) on this page.

t (days)	0	3	7	10	12
$r'(t)$ (centimeters per day)	−6.1	−5.0	−4.4	−3.8	−3.5

Response for question 4(a)

$$\begin{array}{r} \overset{3}{\cancel{4}}.4 \\ 3.8 \\ \hline 0.6 \end{array}$$

$$\frac{-3.8 - -4.4}{10-7} = \frac{0.6}{3} = \frac{\frac{6}{10}}{\frac{3}{1}} = \frac{6}{10} \cdot \frac{1}{3} = \frac{6}{30} = \boxed{\frac{1}{5}\ \text{centimeters}/\text{day}^2}$$

Response for question 4(b)

$-5.0+6.1$

$$\frac{-5.0 - -6.1}{3-0} = \frac{1.1}{3}$$

$$\frac{\frac{11}{10}}{\frac{3}{1}} = \boxed{\frac{11}{30}}$$

NO, the Mean Value theorem does not garentee $r'(t)=-6$ on the interval $0 \le t \le 3$

Answer QUESTION 4 parts (c) and (d) on this page.

Response for question 4(c)

$3(-5.0)+4(-4.4)+3(-3.8)+2(-3.5)$

$-15-17.6-11.4+7=\boxed{-37.0}$

Response for question 4(d)

$\frac{dh}{dt}=-2\text{ cm/day}$

$t=3\text{days}$

$r=100\text{cm}$

$h=50\text{cm}$

$v=\frac{1}{3}\pi r^2 h$

$\frac{dv}{dt}=\frac{2}{3}\pi r\frac{dr}{dt}\cdot h+\frac{dh}{dt}\cdot\frac{1}{3}\pi r^2$

$\frac{2}{3}\pi(100)$

Answer QUESTION 4 parts (a) and (b) on this page.

t (days)	0	3	7	10	12
$r'(t)$ (centimeters per day)	−6.1	−5.0	−4.4	−3.8	−3.5

Response for question 4(a)

$$\frac{-3.8-(-4.4)}{10-7} = \frac{0.6}{3} = \boxed{0.2}$$

Response for question 4(b)

Yes, because the Mean Value Theorem states that if a function is continuous and differentiable, there will be a derived function that can find a value c if $a \le c \le b$. Since −6.0 is between −6.1 and −5.0 in the interval $0 \le t \le 3$, there will be a value for r where $r'(r) = 6$

Answer QUESTION 4 parts (c) and (d) on this page.

Response for question 4(c)

$$\int_0^{12} r'(t)\,dt = 2(-3.5)+3(-3.8)+4(-4.4)+3(-5.0)$$

$$= -7.0-11.4-17.6-15.0 = \boxed{-52.0}$$

Response for question 4(d)

$\frac{dh}{dt} = -2$

$r = 100$ cm

$h = 50$ cm

$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{dV}{dt} = \frac{2}{3}\pi r \frac{dr}{dt} h \frac{dh}{dt}$$

$$\frac{dV}{dt} = \frac{2}{3}\pi(100)(50)(-2)$$

$$\frac{dV}{dt} = \boxed{\frac{-20000}{3}\pi \ \frac{\text{cm}^3}{\text{day}}}$$

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Question 4

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem the melting of an ice sculpture can be modeled as a cone that maintains a conical shape as it decreases in size. The radius of the base of the cone is a twice-differentiable function $r(t)$ measured in centimeters, with time t, $0 \le t \le 12$, in days. Selected values of $r'(t)$ are provided in a table.

In part (a) students were asked to approximate $r''(8.5)$ using the average rate of change of r' over the interval $7 \le t \le 10$ and to provide correct units. A correct response should estimate the value using a difference quotient, drawing from the data in the table that most tightly bounds $t = 8.5.$ The response should include units of centimeters per day per day.

In part (b) students were asked to justify whether there is a time t, $0 \le t \le 3$, for which the rate of change of r is equal to $-6.$ A correct response will use the Intermediate Value Theorem, first noting that the conditions for applying this theorem are met—specifically that $r'(t)$ is continuous because $r(t)$ is twice-differentiable and that -6 is bounded between the values of $r'(0)$ and $r'(3)$ given in the table. Therefore, by the Intermediate Value Theorem, there is a time t such that $0 < t < 3,$ with $r'(t) = -6.$

In part (c) students were asked to use a right Riemann sum and the subintervals indicated by the table to approximate the value of $\int_0^{12} r'(t)\, dt.$ A correct response should present the sum of the four products $\Delta t_i \cdot r'(t_i)$ drawn from the table and obtain an approximation value of $-51.$

In part (d) students were told that the height of the cone decreases at a rate of 2 centimeters per day and that at time $t = 3$ the radius of the cone is 100 cm and the height is 50 cm. They are asked to find the rate of change of the volume of the cone with respect to time at time $t = 3$ days. A correct response will use the product and chain rules to differentiate the given function for the volume of a cone, $V = \frac{1}{3}\pi r^2 h,$ and then evaluate the resulting derivative using values $r = 100,$ $h = 50,$ $\left.\frac{dh}{dt}\right|_{t=3} = -2,$ and $\left.\frac{dr}{dt}\right|_{t=3} = -5$ (from the table) to obtain a rate of $-\frac{70{,}000\pi}{3}$ cubic centimeters per day.

Sample: 4A
Score: 9

The response earned 9 points: 2 points in part (a), 2 points in part (b), 2 points in part (c), and 3 points in part (d).

In part (a) the response would have earned the first point with the expression $\frac{-3.8 - (-4.4)}{3}$ in line 1, with no simplification. In this case, correct simplification to the boxed answer of 0.2 in line 2 earned the first point. The response earned the second point for the correct units presented in the boxed answer in line 2.

Question 4 (continued)

In part (b) the response earned the first point with the statements given in the last three lines, which place the value -6 between the values of -5.0 and -6.1. The response earned the second point with the statements in lines 1 through 5, concluding that $r'(t)$ is continuous because $r'(t)$ is differentiable. The response names the Intermediate Value Theorem, which is not required but is correct.

In part (c) the response earned the first point with the sum of products expression given in line 1. The response would have also earned the second point for this expression in line 1, with no further simplification. In this case, correct simplification to the boxed answer of -51 in line 6 earned the second point.

In part (d) the response earned the first and second points with the correct $\frac{dV}{dt}$ expression given in line 2. The response would have also earned the third point for the correct evaluation of this expression given in line 3, with no further simplification. In this case, correct simplification presented in the boxed answer in line 6 earned the third point.

Sample: 4B
Score: 5

The response earned 5 points: 2 points in part (a), no points in part (b), 1 point in part (c), and 2 points in part (d).

In part (a) the response would have earned the first point with the expression $\frac{-3.8 - -4.4}{10 - 7}$ at the beginning of line 1, with no simplification. In this case, correct simplification to the boxed answer of $\frac{1}{5}$ at the end of line 1 earned the first point. The response earned the second point for the correct units presented in the boxed answer in line 1.

In part (b) the response does not establish that -6 is between $r'(0)$ and $r'(3)$; thus, it did not earn the first point. The response did not earn the second point because the conclusion of "NO" is incorrect.

In part (c) the response earned the first point with the sum of products given in line 1. The response would have also earned the second point for this expression in line 1, with no further simplification. In this case, the response did not earn the second point because of an incorrect simplification to -37.0.

In part (d) the response earned the first and second points with the correct expression for $\frac{dV}{dt}$ given on line 3. The response did not earn the third point because this expression was never evaluated.

Sample: 4C
Score: 3

The response earned 3 points: 1 point in part (a), 1 point in part (b), 1 point in part (c), and no points in part (d).

In part (a) the response would have earned the first point with the expression $\frac{-3.8 - (-4.4)}{10 - 7}$ in line 1, with no simplification. In this case, correct simplification to the final answer of 0.2 at the end of line 1 earned the first point. The response does not present correct units, so it did not earn the second point.

Question 4 (continued)

In part (b) the response earned the first point for the statement given in line 4: "... since -6 is between -6.1 and $-5.0.$" The response does not establish that the continuity condition of the Intermediate Value Theorem has been met and incorrectly names the theorem as the Mean Value Theorem, so it did not earn the second point.

In part (c) the response earned the first point with the sum of products given in line 1. The response would have also earned the second point for this expression in line 1, with no further simplification. In this case, incorrect simplification leads to a final answer of $-52,$ so the response did not earn the second point.

In part (d) the response earned no points. The response does not present either a correct product rule or correct chain rule; thus, it did not earn either of the first two points and is not eligible for the third point.

2022

AP® Calculus BC

Sample Student Responses and Scoring Commentary

Inside:

Free-Response Question 5

☑ Scoring Guidelines

☑ Student Samples

☑ Scoring Commentary

Part B (BC): Graphing calculator not allowed

Question 5 — 9 points

General Scoring Notes

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

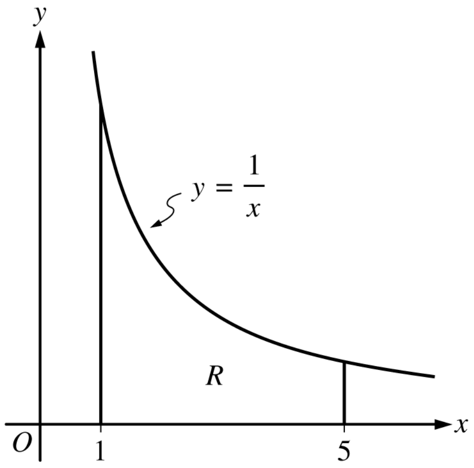


Figure 1

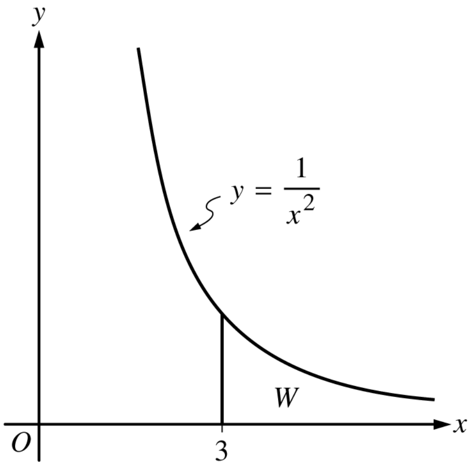


Figure 2

Figures 1 and 2, shown above, illustrate regions in the first quadrant associated with the graphs of $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$, respectively. In Figure 1, let R be the region bounded by the graph of $y = \frac{1}{x}$, the x-axis, and the vertical lines $x = 1$ and $x = 5$. In Figure 2, let W be the unbounded region between the graph of $y = \frac{1}{x^2}$ and the x-axis that lies to the right of the vertical line $x = 3$.

Model Solution	Scoring	
(a) Find the area of region R.		
$\text{Area} = \int_1^5 \frac{1}{x}\,dx$	Integral	**1 point**
$= \ln x \Big\vert_1^5$ $= \ln 5 - \ln 1 = \ln 5$	Answer	**1 point**

Scoring notes:

- A definite integral with incorrect bounds does not earn either point.
- An unevaluated indefinite integral does not earn either point.
- An indefinite integral that is evaluated in a later step may earn one or both points. For example, $\int \frac{1}{x}\,dx = \ln 5 - \ln 1$ (or $\ln 5$) does not earn the first point but does earn the second. However, $\int \frac{1}{x}\,dx = \ln x + C \Rightarrow \text{Area} = \ln 5 - \ln 1$ earns both points.

Total for part (a) 2 points

(b) Region R is the base of a solid. For the solid, at each x the cross section perpendicular to the x-axis is a rectangle with area given by $xe^{x/5}$. Find the volume of the solid.

$\text{Volume} = \int_1^5 xe^{x/5}\,dx$	Definite integral	**1 point**
Using integration by parts, $u = x \qquad dv = e^{x/5}\,dx$ $du = dx \quad v = 5e^{x/5}$	u and dv	**1 point**
$\int xe^{x/5}\,dx = 5xe^{x/5} - \int 5e^{x/5}\,dx$ $= 5xe^{x/5} - 25e^{x/5} + C$ $= 5e^{x/5}(x-5) + C$	$\int xe^{x/5}\,dx$ $= 5xe^{x/5} - \int 5e^{x/5}\,dx$	**1 point**
$\text{Volume} = 5e^{x/5}(x-5)\Big\vert_1^5$ $= 5e(0) - 5e^{1/5}(-4) = 20e^{1/5}$	Answer	**1 point**

Scoring notes:

- The first point is earned for $c\int_1^5 xe^{x/5}\,dx$, where $c \neq 0$. Errors of $c \neq 1$, for example $c = \pi$, will not earn the fourth point.
- Incorrect integrals that require integration by parts are still eligible for the second and third points. Both of these points will be earned with at least one correct application of integration by parts.
- The second point will be earned with an implied u and dv in the presence of $5xe^{x/5} - \int 5e^{x/5}\,dx$.
- The tabular method may be used to show integration by parts. In this case, the second point is earned by having columns (labeled or unlabeled) that begin with x and $e^{x/5}$. The third point is earned for either $5xe^{x/5} - \int 5e^{x/5}\,dx$ or $5xe^{x/5} - 25e^{x/5}$.
- Limits of integration may be present, omitted, or partially present in the work for the second and third points.
- The fourth point is earned only for the correct answer.

Total for part (b) 4 points

(c) Find the volume of the solid generated when the unbounded region W is revolved about the x-axis.

$\text{Volume} = \pi \int_3^{\infty} \left(\frac{1}{x^2}\right)^2 dx = \pi \lim_{b\to\infty} \int_3^b \frac{1}{x^4}\,dx$	Improper integral	**1 point**
$= \pi \lim_{b\to\infty} \left(\left. \frac{1}{-3x^3} \right\|_3^b \right)$	Antiderivative	**1 point**
$= \pi \lim_{b\to\infty} \left(\frac{1}{-3}\right)\left[\frac{1}{b^3} - \frac{1}{3^3}\right]$ $= \pi\left(\frac{1}{-3}\right)\left(0 - \frac{1}{3^3}\right) = \frac{\pi}{81}$	Answer	**1 point**

Scoring notes:

- The first point is earned for either $c\int_3^{\infty} \left(\frac{1}{x^2}\right)^2 dx$ or $\lim_{b\to\infty} c\int_3^b \frac{1}{x^4}\,dx,$ where $c \neq 0.$ Errors of $c \neq \pi$ will not earn the third point.
- The second point is earned for a correct antiderivative of any integrand of the form $\frac{1}{x^n},$ for any integer $n \geq 2.$
- To earn the answer point, a response must use correct limit notation and cannot include arithmetic with infinity, such as $\frac{1}{\infty^3}.$

Total for part (c) 3 points

Total for question 5 9 points

Answer QUESTION 5 part (a) on this page.

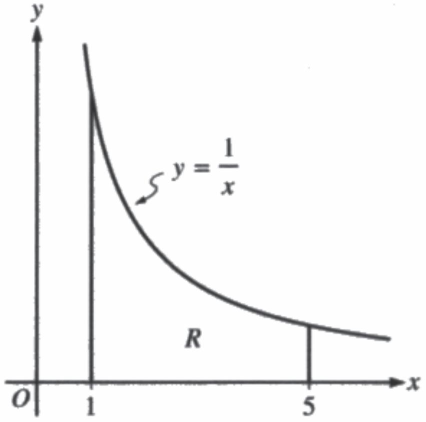


Figure 1

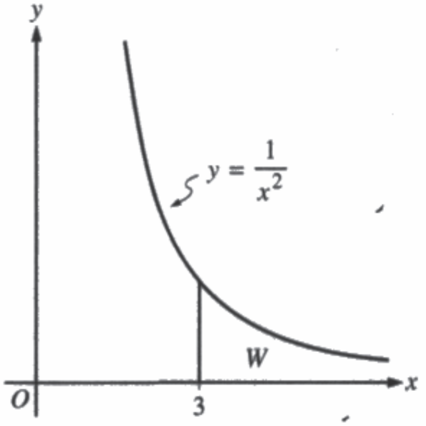


Figure 2

Response for question 5(a)

(a) area of R = $\int_1^5 \frac{1}{x}\,dx = [\ln|x|]_1^5 = \ln 5 - \ln 1 = \boxed{\ln 5}$

The area of region R is ln 5.

Answer QUESTION 5 parts (b) and (c) on this page.

Response for question 5(b)

$dV = xe^{x/5}\,dx$

$V = \int_1^5 xe^{x/5}\,dx$

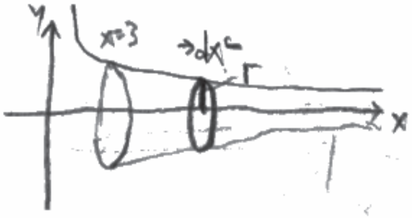
$u = x \quad dv = e^{x/5}\,dx$

$du = dx \quad v = 5e^{x/5}$

$V = \left[5xe^{x/5}\right]_1^5 - \int_1^5 5e^{x/5}\,dx = (25e - 5e^{1/5}) - \left[25e^{x/5}\right]_1^5$

$= 25e - 5e^{1/5} - (25e - 25e^{1/5}) = \boxed{20e^{1/5}}$

Response for question 5(c)



$dV = \pi r^2\,dx$

$= \pi\left(\frac{1}{x^2}\right)^2 dx$

$V = \int_3^\infty \pi\left(\frac{1}{x^2}\right)^2 dx$

$= \pi \lim_{t\to\infty} \int_3^t \frac{1}{x^4}\,dx$

$= \pi \lim_{t\to\infty} \left[\frac{x^{-3}}{-3}\right]_3^t$

$= -\frac{\pi}{3} \lim_{t\to\infty} \left(\frac{1}{t^3} - \frac{1}{27}\right)$

$= \frac{\pi}{3}\left(\frac{1}{27}\right) = \boxed{\frac{\pi}{81}}$

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Answer QUESTION 5 part (a) on this page.

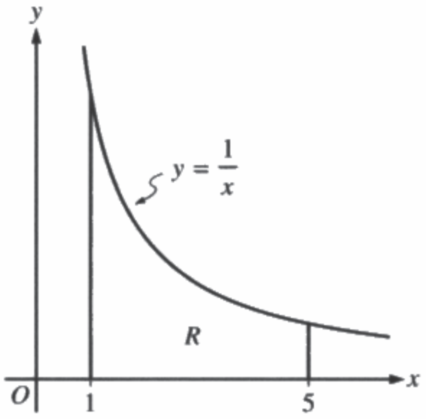


Figure 1

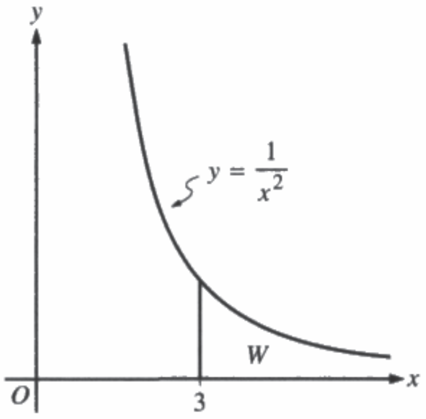


Figure 2

Response for question 5(a)

$$\int_1^5 \frac{1}{x}\,dx = \ln x\Big]_1^5$$

$$= \boxed{\ln 5} - \underset{=0}{\cancel{\ln 1}}$$

Answer QUESTION 5 parts (b) and (c) on this page.

Response for question 5(b)

$$\int_1^5 xe^{\frac{x}{5}}\,dx$$

$u = x \qquad v = 5e^{\frac{x}{5}}$

$du = dx \qquad dv = e^{\frac{x}{5}}dx$

$$5xe^{\frac{x}{5}}\Big]_1^5 - \int_1^5 5e^{\frac{x}{5}}\,dx$$

$$5xe^{\frac{x}{5}}\Big]_1^5 \quad - 25e^{\frac{x}{5}}\Big]_1^5$$

$$\boxed{\left(5(5)e - 5e^{\frac{1}{5}}\right) - \left(25e - 25e^{\frac{1}{5}}\right)}$$

Response for question 5(c)

$$\int_3^\infty \frac{1}{x^2}\,dx$$

$$\lim_{b\to\infty} \int_3^b \frac{1}{x^2}\,dx$$

$$= \lim_{b\to\infty}\left(-\frac{1}{x}\right]_3^b$$

$$= \lim_{b\to\infty}\left(-\frac{1}{b} + \frac{1}{3}\right)$$

$$= \boxed{\frac{1}{3}}$$

Answer QUESTION 5 part (a) on this page.

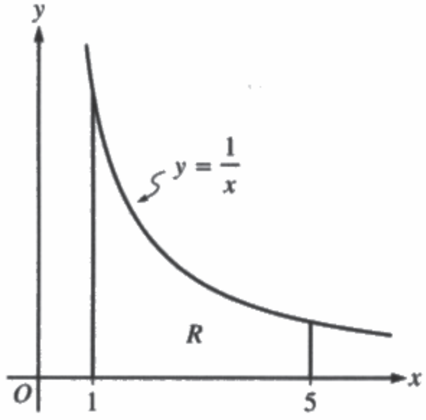


Figure 1

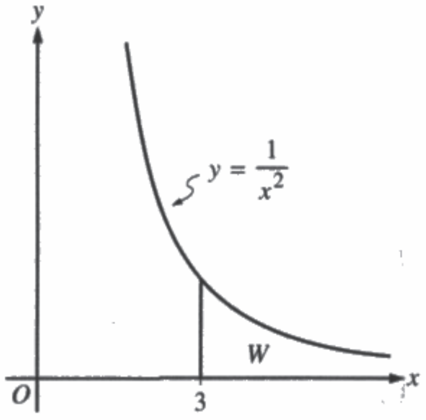


Figure 2

Response for question 5(a)

Area of region R

$\int_1^5 \frac{1}{x} dx$ $\quad$ $u = x$ $\quad$ $du = 1$

$= \int_1^5 \frac{du}{u}$

$\ln x \Big|_1^5$

$\ln 5 - \ln 1$

$\boxed{= \ln 4}$

5 5 5 5 5 NO CALCULATOR ALLOWED 5 5 5 5 5

Answer QUESTION 5 parts (b) and (c) on this page.

Response for question 5(b)

$V = \int A$

$A = S^2$

$V = \int_1^5 x e^{x/5} dx$

$= \frac{x^2}{5} e^{x/5} \Big|_1^5$

$= \left(\frac{25}{5} e^{5/5}\right) - \left(\frac{1}{5} e^{1/5}\right)$

$= (5e) - \frac{1}{5} e^{1/5}$

Response for question 5(c)

$V = \int_3^{10} \left(\frac{1}{x^2} - 0\right)^2 dx$

$V = \int_3^0 \frac{1}{x^4} dx$

$4 \ln x \Big|_3^0$

$\boxed{4 - 4\ln 3}$

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Question 5

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

This problem provided two figures, illustrating regions R and W in the first quadrant, associated with the graphs of $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$, respectively.

In part (a) students were asked to find the area of region R. A correct response would recognize the need to compute the definite integral of $y = \frac{1}{x}$ from $x = 1$ to $x = 5$, find an antiderivative of $\ln x$, and present an area of $\ln 5$.

In part (b) students were asked to find the volume of a solid that has region R as its base and whose cross sections perpendicular to the x-axis are rectangles with area $xe^{x/5}$. A correct response would integrate the given area function $y = xe^{x/5}$ from $x = 1$ to $x = 5$ and then proceed to use integration by parts to evaluate the integral.

In part (c) students were asked to find the volume of the solid generated when the unbounded region W is revolved about the x-axis. A correct response would recognize this volume as π times the improper integral of the square of the function $y = \frac{1}{x^2}$, starting at $x = 3$. The response would use correct limit notation to rewrite the improper integral with a variable upper limit, find a correct antiderivative $\left(\int \frac{1}{x^4}\,dx = -\frac{1}{3x^3} + C\right)$, and continue the correct limit notation to find a volume of $\frac{\pi}{81}$.

Sample: 5A
Score: 9

The response earned 9 points: 2 points in part (a), 4 points in part (b), and 3 points in part (c).

In part (a) the response earned the first point with the correct definite integral $\int_1^5 \frac{1}{x}\,dx$ in line 1. The response would have earned the second point with $\ln 5 - \ln 1$ in line 1. In this case, the response correctly simplifies to the boxed answer of $\ln 5$ at the end of line 1 and earned the second point. The response confirms this answer with the sentence in line 2.

In part (b) the response earned the first point with the correct definite integral $\int_1^5 xe^{x/5}\,dx$ in line 2. The response earned the second point by correctly identifying u and dv on line 3. The response correctly applies integration by parts with the expression $\left[5xe^{x/5}\right]_1^5 - \int_1^5 5e^{x/5}\,dx$ in line 5 and earned the third point. The expression at the start of line 6 would have earned the fourth point with no simplification. In this case, simplification to the boxed answer of $20e^{1/5}$ at the end of line 6 is correct, and the fourth point was earned.

Question 5 (continued)

In part (c) the response earned the first point on line 3 with the improper integral $\int_3^\infty \pi\left(\frac{1}{x^2}\right)^2 dx.$ The response earned the second point with the correct antiderivative $\frac{x^{-3}}{-3}$ in line 5. The response earned the third point with the boxed answer $\frac{\pi}{81}$ at the end of line 7 and the correct use of limit notation in lines 4 through 7.

Sample: 5B
Score: 7

The response earned 7 points: 2 points in part (a), 4 points in part (b), and 1 point in part (c).

In part (a) the response earned the first point with the correct definite integral $\int_1^5 \frac{1}{x}\,dx$ in line 1. The response earned the second point with the circled answer of $\ln 5$ in line 2.

In part (b) the response earned the first point with the correct definite integral $\int_1^5 xe^{\frac{x}{5}}\,dx$ in line 1. The response correctly identifies u and dv in the side work to the right and earned the second point. The response earned the third point with the correct application of integration by parts, $5xe^{\frac{x}{5}}\Big|_1^5 - \int_1^5 5e^{\frac{x}{5}}\,dx,$ in line 2. The response earned the fourth point with the boxed answer in line 4. Numerical simplification is not required.

In part (c) the response did not earn the first point in line 1 with the incorrect improper integral $\int_3^\infty \frac{1}{x^2}\,dx.$ The response is eligible to earn the second point because the integrand is of the form $\frac{1}{x^n},\ n > 1.$ The response earned the second point with $-\frac{1}{x},$ the correct antiderivative of $\frac{1}{x^2}.$ The response did not earn the third point because the answer is not correct.

Sample: 5C
Score: 2

The response earned 2 points: 1 point in part (a), 1 point in part (b), and no points in part (c).

In part (a) the response earned the first point with the correct definite integral $\int_1^5 \frac{1}{x}\,dx$ in line 2. The response would have earned the second point with the evaluation $\ln 5 - \ln 1$ in line 5. In this case, there is a simplification error that results in an incorrect boxed answer in line 6. Therefore, the second point was not earned.

In part (b) the response earned the first point with the correct definite integral $V = \int_1^5 xe^{x/5}\,dx$ in line 3. The response did not earn the second, third, and fourth points because the response does not use integration by parts, and the answer is not correct.

Question 5 (continued)

In part (c) the response did not earn the first point because the improper integral in line 1 has limits of integration that are not consistent with region W. The response is eligible to earn the second point because the integrand is of the form $\frac{1}{x^n}$, $n > 1$. The response did not earn the second point because the antiderivative in line 3 is not correct. The response did not earn the third point because the boxed answer is not correct.

2022

AP® Calculus BC

Sample Student Responses and Scoring Commentary

Inside:

Free-Response Question 6

☑ Scoring Guidelines

☑ Student Samples

☑ Scoring Commentary

Part B (BC): Graphing calculator not allowed
Question 6 **9 points**

General Scoring Notes
The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

The function f is defined by the power series $f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$ for all real numbers x for which the series converges.

Model Solution	Scoring	
(a) Using the ratio test, find the interval of convergence of the power series for f. Justify your answer.		
$\lim\limits_{n\to\infty}\left\lvert \dfrac{\frac{(-1)^{n+1}x^{2n+3}}{2n+3}}{\frac{(-1)^n x^{2n+1}}{2n+1}} \right\rvert = \lim\limits_{n\to\infty}\left\lvert \dfrac{\frac{x^{2n+3}}{2n+3}}{\frac{x^{2n+1}}{2n+1}} \right\rvert = \lim\limits_{n\to\infty}\left\lvert x^2\left(\frac{2n+1}{2n+3}\right)\right\rvert = \lvert x^2\rvert$	Sets up ratio	**1 point**
$\lvert x^2\rvert < 1$ for $\lvert x\rvert < 1$. The series converges when $-1 < x < 1$.	Identifies interior of interval of convergence	**1 point**
When $x = -1$, the series is $-1 + \frac{1}{3} - \frac{1}{5} + \cdots + \frac{(-1)^{n+1}}{2n+1} + \cdots$. The series is an alternating series whose terms decrease in absolute value to 0. The series converges by the Alternating Series Test. When $x = 1$, the series is $1 - \frac{1}{3} + \frac{1}{5} + \cdots + \frac{(-1)^n}{2n+1} + \cdots$. The series is an alternating series whose terms decrease in absolute value to 0. The series converges by the Alternating Series Test. Therefore, the interval of convergence is $-1 \le x \le 1$.	Considers both endpoints	**1 point**
	Analysis and interval of convergence	**1 point**

Scoring notes:

- A response that includes the substitution error of the form $x^{2(n+1)+1} = x^{2n+3}$ appearing as $x^{2n+1+1} = x^{2n+2}$ in setting up a ratio is eligible for the first 3 points but does not earn the fourth point.
- The first point is earned by presenting a correct ratio with or without absolute values.
- To earn the second point a response must:
  - o use the absolute value of the ratio, or resolve the lack of absolute values by concluding $x^2 < 1$ (without any errors), and correctly evaluate the limit of the ratio, including correct limit notation, and
  - o identify the interior of the interval of convergence. The response can use either interval notation or the compound inequality $-1 < x < 1$ ($|x| < 1$ is insufficient).
- The only incorrect interval eligible for the third point is $0 < x < 1$. In this case, to earn the third point, the response needs to evaluate the general term at $x = 1$.

Total for part (a) 4 points

(b) Show that $\left|f\left(\frac{1}{2}\right) - \frac{1}{2}\right| < \frac{1}{10}$. Justify your answer.

The series for $f\left(\frac{1}{2}\right)$ is an alternating series whose terms decrease in absolute value to 0. The first term of the series for $f\left(\frac{1}{2}\right)$ is $\frac{1}{2}$. Using the alternating series error bound, $f\left(\frac{1}{2}\right)$ differs from $\frac{1}{2}$ by at most the absolute value of the second term of the series.

$$\left|f\left(\frac{1}{2}\right) - \frac{1}{2}\right| < \left|\frac{(-1)^1\left(\frac{1}{2}\right)^3}{3}\right| = \frac{1}{24} < \frac{1}{10}$$

Uses second term	**1 point**
Justification	**1 point**

Scoring notes:

- The first point is earned by correctly using $x = \frac{1}{2}$ in the second term (listing the second term as part of a polynomial is insufficient). Using $x = \frac{1}{2}$ in any term of degree five or higher does not earn this point.
- To earn the second point a response must:
  - o have earned the first point,
  - o state that the series is alternating and that its terms decrease to zero, and
  - o present the inequality $\text{Error} < \frac{1}{24} < \frac{1}{10}$ (or the equivalent).
- A response that states $\text{Error} = \frac{1}{24}$ does not earn the second point.

Total for part (b) 2 points

(c) Write the first four nonzero terms and the general term for an infinite series that represents $f'(x)$.

$f'(x) = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots$	First four terms	**1 point**
	General term	**1 point**

Scoring notes:

- The first point is earned by presenting the first four nonzero terms in a list or as part of a polynomial or series.
- The second point is earned by identifying the general term (either individually or as part of a polynomial or series).
- Read "$=$" as "$\approx$" as necessary.

Total for part (c) **2 points**

(d) Use the result from part (c) to find the value of $f'\left(\frac{1}{6}\right)$.

$f'\left(\frac{1}{6}\right) = 1 - \left(\frac{1}{6}\right)^2 + \left(\frac{1}{6}\right)^4 - \left(\frac{1}{6}\right)^6 + \cdots$ $f'\left(\frac{1}{6}\right)$ is a geometric series with $a = 1$ and $r = -\frac{1}{36}$. $f'\left(\frac{1}{6}\right) = \frac{a}{1-r} = \frac{1}{1-\left(-\frac{1}{36}\right)} = \frac{1}{\frac{37}{36}} = \frac{36}{37}$	Answer	**1 point**

Scoring notes:

- The result from part (c) must be geometric in order to be eligible for this point.
- If a response imports an incorrect geometric series from part (c), this point is earned only for a consistent answer.

Total for part (d) **1 point**

Total for question 6 **9 points**

Answer QUESTION 6 parts (a) and (b) on this page.

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

Response for question 6(a)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1} x^{2n+3}}{2n+3} \cdot \frac{2n+1}{(-1)^n x^{2n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^{2n+3}}{x^{2n+1}} \cdot \frac{2n+1}{2n+3}\right| = \left|x^2\right| < 1$$

IOC:

$-1 \le x \le 1$

ROC $= -1 < x < 1$

@ -1

$$\sum_{0}^{\infty} \frac{(-1)^n \cdot (-1)^{2n+1}}{2n+1} = \frac{(-1)^{3n+1}}{2n+1}$$ converges by AST

@ 1

$$\sum_{0}^{\infty} (-1)^n \cdot \frac{1^{2n+1}}{2n+1} = \frac{(-1)^n}{2n+1}$$ converges by AST

Response for question 6(b)

$R_n \le |a_{n2}|$

$R_n \le \left|\frac{x^3}{3}\right|$ @ $x=\frac{1}{2}$

$R_n \le \left|\frac{1}{8}\cdot\frac{1}{3}\right| = \left|\frac{1}{24}\right| < \frac{1}{10}$

The remainder ~~is the~~ less than equal to the next unused term which is the $-\frac{x^3}{3}$ term.

@ $x=\frac{1}{2}$

$\left|-\frac{x^3}{3}\right|$ is $\frac{1}{24}$

which is less than $\frac{1}{10}$

Answer QUESTION 6 parts (c) and (d) on this page.

Response for question 6(c)

$$f'(x) = 1 - x^2 + x^4 - x^6$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

Response for question 6(d)

$$f'\left(\tfrac{1}{6}\right) = 1 - \left(\frac{1}{6}\right)^2 + \left(\frac{1}{6}\right)^4 - \left(\frac{1}{6}\right)^6$$

Answer QUESTION 6 parts (a) and (b) on this page.

Response for question 6(a)

$\frac{(-1)^n x^{2n+1}}{2n+1}$ (absolute value → no $(-1)^n$)

$\lim_{x\to\infty} \left| \frac{x^{2(n+1)+1}}{2(n+1)+1} \cdot \frac{2n+1}{x^{2n+1}} \right| < 1$

$\lim_{x\to\infty} \left| x^2 \cdot \frac{2n+1}{2n+3} \right| < 1$

$|x^2| < 1$

$0 < \underbrace{x^2}_{\text{always} > 0} < 1$

$0 < x < 1$

→ test bounds

$x = 1$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$

converges by alternating series test

$x = 0$

→ $\sum_{n=1}^{\infty} 0$ → converges to 0

$\boxed{[0, 1]}$

Response for question 6(b)

$|f(\frac{1}{2}) - \frac{1}{2}| < \frac{1}{10}$

$|f(\frac{1}{2}) - \frac{1}{2}| < a_{n+1}$ (or next term)

$a_{n+1} = \frac{x^3}{3}$

$a_{n+1} = \frac{(\frac{1}{2})^3}{3}$

$a_{n+1} = \frac{1}{8(3)} = \frac{1}{24} < \frac{1}{10}$ ✓

Answer QUESTION 6 parts (c) and (d) on this page.

Response for question 6(c)

$$f'(x) = 1 - x^2 + x^5 - x^7 + \ldots + \frac{(-1)^n (2n+1) x^{2n}}{(2n+1)} + \ldots$$

$$f'(x) = 1 - x^2 + x^5 - x^7 + \ldots + (-1)^n x^{2n} + \ldots$$

Response for question 6(d)

$$f'\left(\tfrac{1}{6}\right) \approx 1 - \left(\tfrac{1}{6}\right)^2 + \left(\tfrac{1}{6}\right)^5 - \left(\tfrac{1}{6}\right)^7$$

$$f'\left(\tfrac{1}{6}\right) \approx 1 - \frac{1}{36} + \frac{1}{6^5} - \frac{1}{6^7}$$

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Answer QUESTION 6 parts (a) and (b) on this page.

Response for question 6(a)

① Ratio Test

$$\sum \frac{(-1)^n x^{2n+1}}{(2n+1)}$$

$$\left| \frac{x^{(2(n+1)+1)}}{(2(n+1)+1)} \cdot \frac{2n+1}{x^{2n+1}} \right| < 1$$

$$\left| \frac{\cancel{(x^{2n})}^{+1+2}}{\cancel{2n+1}+2} \cdot \frac{\cancel{(2n+1)}}{\cancel{(x^{2n})}^{+1}} \right| < 1$$

$$\left| \frac{x^{2n+3}}{x^{2n+1}} \right| < 1$$

$|x^2| < \textcircled{1}$ ROC

$-1 < x < 1$

② Test pts

$x = -1$

$$\sum \frac{(-1)^{-1}(x)^{\overset{-1}{2(-1)+1}}}{(2(-1)+1)} = \frac{1}{(-1)(x^{-1})(2(-1)+1)} \sum \frac{1}{x}$$

so div (comp to $\sum \frac{1}{n}$ harmonic p series)

$x = 1$

$$\sum \frac{(-1)^1 (x)^{2(1)+1}}{(2(1)+1)} = x \dots$$

goes to ∞

$$\boxed{-1 < x < 1}$$

Response for question 6(b)

Lagrange

$$\frac{f^{n+1}(\max)}{(n+1)!} (x+c)^{n+1}$$

$$\frac{f^{(8)}(.8)}{8!}(x-\tfrac{1}{2})^8 < \frac{1}{10}$$

$$\frac{\left(\frac{1}{2}\right)^9}{9} < \frac{1}{10}$$

$f^{(8)}$

$$f^8(8) = \frac{(-1)^8(x^{2(8)+1})}{2(8)+1}$$

$$\frac{\cancel{(-1)^8}(x^{17})}{17}$$

$$f^{(4)} = \frac{\cancel{(-1)^4}(x^{2(4)+1})}{2(4)+1} = \boxed{\frac{x^9}{9}}$$

max error

a_{n+1} term

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Answer QUESTION 6 parts (c) and (d) on this page.

Response for question 6(c)

Find f'

$$f = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots + \frac{(-1)^n x^{2n+1}}{2n+1}$$

$$f' = 1 - \frac{\cancel{3}x^2}{\cancel{3}} + \frac{\cancel{5}x^4}{\cancel{5}} - \frac{\cancel{7}x^6}{\cancel{7}}$$

$$f' = 1 - x^2 + x^4 - x^6 + \ldots + \frac{n(-1)^{n-1}x^{(2n)}}{(2n)}$$

Response for question 6(d)

$$f'\left(\frac{1}{6}\right) = 1 - \left(\frac{1}{6}\right)^2 + \left(\frac{1}{6}\right)^4 - \left(\frac{1}{6}\right)^6$$

Question 6

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem a function f is defined by a power series for all real numbers for which the power series converges.

In part (a) students were asked to use the ratio test to find the interval of convergence for the power series. A correct response should set up the ratio $\left|\frac{a_{n+1}}{a_n}\right|$ and find the limit of this ratio as $n \to \infty$ to find the interior of the interval of convergence. The response should then use the Alternating Series Test to determine that the series converges for both endpoints of this interval of convergence.

In part (b) students were asked to justify that $\left|f\left(\frac{1}{2}\right) - \frac{1}{2}\right| < \frac{1}{10}$. A correct response would recognize that $\frac{1}{2}$ is the first term of the series for $f\left(\frac{1}{2}\right)$ and that the series for f is alternating with terms that decrease in absolute value to 0. Therefore, $\left|f\left(\frac{1}{2}\right) - \frac{1}{2}\right|$ must be no more than the second term of the series for $f\left(\frac{1}{2}\right)$, which is $\frac{1}{24} < \frac{1}{10}$.

In part (c) students were asked to write the first four terms and the general term for an infinite series that represents $f'(x)$. A correct response would differentiate the first four given terms of the series for $f(x)$.

Finally, in part (d) students were asked to use the series from part (c) to find the value of $f'\left(\frac{1}{6}\right)$. A correct response must recognize that the series for $f'\left(\frac{1}{6}\right)$ is geometric with $a = 1$ and $r = -\frac{1}{36}$. The value of $f'\left(\frac{1}{6}\right)$ is the sum of the geometric series, $\frac{36}{37}$.

Sample: 6A
Score: 7

The response earned 7 points: 4 points in part (a), 1 point in part (b), 2 points in part (c), and no points in part (d).

In part (a) the response earned the first point by presenting the correct ratio. The response earned the second point by presenting the correct interior of the interval of convergence with the correct limit notation. The response earned the third point by evaluating the general term at $x = -1$ and $x = 1$. The response earned the fourth point by stating "converges by AST" and presenting the correct interval of convergence.

In part (b) the response earned the first point in the third line by using the second term of the series evaluated at $\frac{1}{2}$. The response did not earn the second point because there is no reference to this as an alternating series whose terms decrease to 0.

In part (c) the response earned both the first point and the second point by presenting the correct terms.

In part (d) the response does not present the correct answer and did not earn the point.

Question 6 (continued)

Sample: 6B
Score: 4

The response earned 4 points: 2 points in part (a), 1 point in part (b), 1 point in part (c), and no points in part (d).

In part (a) the response earned the first point in the second line by presenting the correct ratio. The response did not earn the second point because an incorrect interval is presented. The response presents the only other eligible interval of $0 < x < 1$ and, therefore, is eligible for the third point. The response earned the third point by evaluating the general term at $x = 1$. The response did not earn the fourth point because the interval of convergence is incorrect.

In part (b) the response earned the first point by using the second term evaluated at $\frac{1}{2}$. The response did not earn the second point because there is no reference to this as an alternating series whose terms decrease to 0.

In part (c) the response did not earn the first point because the terms are not all correct. The response earned the second point because the general term is correct.

In part (d) the response does not present the correct answer and did not earn the point.

Sample: 6C
Score: 2

The response earned 2 points: 1 point in part (a), no points in part (b), 1 point in part (c), and no points in part (d).

In part (a) the response earned the first point by presenting the correct ratio. The response did not earn the second point because there is no limit notation. The response did not earn the third point because the general term is incorrectly evaluated at both $x = 1$ and $x = -1$. The response did not earn the fourth point because the interval of convergence is incorrect.

In part (b) the response did not earn the first point because the incorrect term is used. The response is not eligible for the second point.

In part (c) the response earned the first point by presenting the correct first four nonzero terms. The response did not earn the second point because the general term is incorrect.

In part (d) the response does not present the correct answer and did not earn the point.